Can numbers be lonely? And how many are such?

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March 16, 2017

Abstract

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- Recently, in [1], Loomis has brought up new sub-classes of solitary numbers.
- In this presentation, we will proceed in proving the density of these sub-classes of solitary numbers are indeed all zero.
- This result helped us one step further on the way of proving the conjecture.

Overview

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- Our main result
- The proof of the main theorem

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Example

$\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28.$

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$A(12) = \frac{\sigma(12)}{12} = \frac{28}{12} = \frac{7}{3}$.

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6 is a friend of 28, because $A(6) = \frac{\sigma(6)}{6} = \frac{1+2+3+6}{6} = 2$, and $A(28) = \frac{\sigma(28)}{28} = \frac{1+2+4+7+14+28}{28} = 2$. Thus $A(6) = A(28) = 2$.

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Example

$2, 3, 5, 7, 9, 11....$ are all lonely numbers. :(

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3. As an example, we do **not yet** know whether 10 is solitary or not.

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If $\gcd(n, \sigma(n)) = 1$, then n is solitary.

Example

9 is solitary, since $\sigma(9) = 1 + 3 + 9 = 13$, and so
$\gcd(9, \sigma(9)) = \gcd(9, 13) = 1$

one subclass of solitary numbers

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Let n be an odd natural numbers.
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Then n^2 *is solitary.*

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Example

Here are some square free numbers:
$6 = 2.3, 30 = 2.3.5, 210 = 2.3.5.7$ and much more :D

New known solitary numbers from this subclass

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From Loomis's Theorem, we can find these new solitary numbers:

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$21^2, 39^2, 57^2, 63^2, 77^2, 93^2$ are the smallest new small known solitary numbers obtained from this theorem.
For detailed computation, take 21^2 as an example. We have $\gcd(21^2, \sigma(21^2)) = \gcd(21^2, 741) = 3$, and 3 is square free. Thus by Loomis's Theorem, $21^2 = 441$ is solitary.

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There is nothing more than high school algebra level in the proof.

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2. Here for example, we could ask ourselves: "How dense the set of solitary numbers is, in the set of Natural numbers?"
3. But what do we really mean by density? And in fact, there are a lot of types of density. The one we will study mainly called **natural density**.

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4. So... what should we do?

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3. Let P_m denote the probability that the chosen number n was solitary. Note that this probability depends on m.
4. Taking m going to infinity, that quantity is the **Natural Density** of the set of Solitary Numbers in the set of positive natural numbers $\mathbb{N}$. that is,

$$\lim_{m \to \infty} P_m$$

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$$\lim_{x \to \infty} \frac{\mathscr{A}(x)}{x}.$$

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Remark

The second exercise is one of the reason why we suspected that the Natural Density of the set of solitary number is zero,

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Answer: Zero. Again, the proof is not easy.

Remark

The second exercise is one of the reason why we suspected that the Natural Density of the set of solitary number is zero, because the set of all prime numbers is a subclass of the set of all solitary numbers.

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$$f(x) \ll g(x) \text{ or } f(x) = \mathcal{O}(g(x))$$

if there is a positive constant c such that

$$|f(x)| < cg(x)$$

for all sufficiently large x.

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Then $\mathscr{A}$ has **natural density** *zero.*

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2. So let's begin, shall we :D ?

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2. Then **by theory**, there exits an absolute constant c_x such that $\sigma(n^2)$ is divisible by all prime p satisfying

$$p < c_x \frac{\log \log x}{\log \log \log x}$$

for all $n^2 < x$, except for a subset of $n^2 \in \mathbb{N}$ of cardinality $\mathcal{O}(\frac{x}{\log \log \log x})$.

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2. what does that mean?
3. If we denote the constant $y_x = c_x \frac{\log \log x}{\log \log \log x}$, then the above is being rewritten in a less wordy way:

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2. still too much to understand? Yes :(. Let's break it into an easier way.

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4. But what does the part "except for a subset of $n^2 \in \mathbb{N}$ of cardinality $\mathcal{O}(\frac{x}{\log \log \log x})$ " mean?

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2. The word **almost every time** already suggested something.
3. How many n^2 that we picked? Says we pick z_x such natural numbers possible.

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2. Then among these z_x natural numbers, there are a small amount of number of n^2 that does not have the property "all the primes $p < y_x$ is divisible by it". This small amount is less than $D_x = \frac{x}{\log \log \log x}$.

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3. when x goes to infinity, D_x goes to zero.

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3. when x goes to infinity, D_x goes to zero.
4. Hence the **small** amount is indeed very small.

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Proof:(continued)

1. So we can say that **almost every** $n^2 < x$ that we picked, they have the properties that
$$p \mid \sigma(n^2)$$
for all prime $p < y_x$.

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1. So we can say that **almost every** $n^2 < x$ that we picked, they have the properties that
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for all prime $p < y_x$.
2. The exception is very small when x is getting very big, hence **can be ignore**.

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2. we denote $\mathscr{A}_x := \mathscr{A} \cap [1, x]$ and
3. $$\mathcal{B}_x := \{n^2 \in \mathbb{N} : (n^2, p) = 1 \text{ for all } p \leq y_x \text{ and } p \mid \sigma(n^2)\}$$

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4. **Main Claim:**
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Proof:(continued) We now proceed in proving the main claim with elementary method:

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2. We proceed by using the proof by contradiction argument.
3. Let $n^2 \in \mathscr{A}_x$
4. And suppose that $\gcd(n^2, p) > 1$ for some $p \leq y_x$, p is prime and $p \mid \sigma(n^2)$.

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Proof:(continued) We now proceed in proving the main claim with elementary method:

1. **Main Claim:**

$$\mathscr{A}_x \subset \mathcal{B}_x := \{n^2 \in \mathbb{N} : (n^2, p) = 1 \text{ for all } p \leq y_x \text{ and } p \mid \sigma(n^2)\}$$

2. We proceed by using the proof by contradiction argument.
3. Let $n^2 \in \mathscr{A}_x$
4. And suppose that $\gcd(n^2, p) > 1$ for some $p \leq y_x$, p is prime and $p \mid \sigma(n^2)$.
5. Note that $\gcd(n^2, p) \mid p$, and so $\gcd(n^2, p) \in \{1, p\}$. Since $\gcd(n^2, p) > 1$ by our assumption, it must be the case that $(n^2, p) = p$. Then $p \mid n^2$. So $p \mid n$. Hence $p^2 \mid n^2$.

Proof of Main Result

Proof:(continued)

1. Now since $p \mid n^2$ and from the beginning of the claim's proof, $p \mid \sigma(n^2)$, so $p \mid \gcd(n^2, \sigma(n^2))$.

Proof of Main Result

Proof:(continued)

1. Now since $p \mid n^2$ and from the beginning of the claim's proof, $p \mid \sigma(n^2)$, so $p \mid \gcd(n^2, \sigma(n^2))$.
2. Now since $n^2 \in \mathscr{A}_x$, so $(n^2, \sigma(n^2))$ is square free, so we can write $L = \gcd(n^2, \sigma(n^2)) = p_1 p_2 \dots p_k$ for some distinct primes $p_1, p_2, \dots, p_k$.

Proof of Main Result

Proof:(continued)

1. Now since $p \mid n^2$ and from the beginning of the claim's proof, $p \mid \sigma(n^2)$, so $p \mid \gcd(n^2, \sigma(n^2))$.
2. Now since $n^2 \in \mathscr{A}_x$, so $(n^2, \sigma(n^2))$ is square free, so we can write $L = \gcd(n^2, \sigma(n^2)) = p_1 p_2 \ldots p_k$ for some distinct primes $p_1, p_2, \ldots, p_k$.
3. Note that the powers of each of these primes p_i in the prime decomposition of $L = \gcd(n^2, \sigma(n^2))$ must be one.Thus $p \mid p_1 p_2 \ldots p_k$.

Proof of Main Result

Proof:(continued)

1. Now since $p \mid n^2$ and from the beginning of the claim's proof, $p \mid \sigma(n^2)$, so $p \mid \gcd(n^2, \sigma(n^2))$.
2. Now since $n^2 \in \mathscr{A}_x$, so $(n^2, \sigma(n^2))$ is square free, so we can write $L = \gcd(n^2, \sigma(n^2)) = p_1 p_2 \ldots p_k$ for some distinct primes $p_1, p_2, \ldots, p_k$.
3. Note that the powers of each of these primes p_i in the prime decomposition of $L = \gcd(n^2, \sigma(n^2))$ must be one.Thus $p \mid p_1 p_2 \ldots p_k$.
4. Since all $p, p_1, \ldots, p_k$ are prime, it must be the case that $p = p_i$ for some i. Without loss of generality, says $p = p_1$.

Proof of Main Result

Proof:(continued)

1. Then $L = \gcd(\sigma(n^2), n^2) = pp_2p_3\ldots p_k$.
2. So we can write $n^2 = pp_2\ldots p_kM$, where M satisfies $\gcd(M, pp_2, \ldots, p_k) = 1$.
3. Then by $p^2 \mid n^2 = pp_2\ldots p_kM$, we deduce $p \mid p_2\ldots p_kM$. Then $p \mid p_j$ for some $j = 2, 3, \ldots, k$ or $p \mid M$.

Proof of Main Result

Proof:(continued)

1. In the first case, the case that $p \mid p_j$ for some $j = 2, 3, \ldots, k$, without loss of generality, says $p \mid p_2$.

Proof of Main Result

Proof:(continued)

1. In the first case, the case that $p \mid p_j$ for some $j = 2, 3, \ldots, k$, without loss of generality, says $p \mid p_2$.
2. Then we can write $p_2 = pD$ for some positive integer D.

Proof of Main Result

Proof:(continued)

1. In the first case, the case that $p \mid p_j$ for some $j = 2, 3, \ldots, k$, without loss of generality, says $p \mid p_2$.
2. Then we can write $p_2 = pD$ for some positive integer D.
3. Then $L = pp_2p_3 \ldots p_k = ppDp_3 \ldots p_k = p^2S$, for some positive integer $S = Dp_3 \ldots p_k$, this contradicts L being square free.

Proof of Main Result

Proof:(continued)

1. In the latter case when $p \mid M$, then $(M, pp_2p_3 \ldots, p_k) \geq p$, contradicts $(M, pp_2, \ldots, p_k) = 1$.

Proof of Main Result

Proof:(continued)

1. In the latter case when $p \mid M$, then $(M, pp_2p_3 \ldots, p_k) \geq p$, contradicts $(M, pp_2, \ldots, p_k) = 1$.
2. In both cases, we get a contradiction.

Proof of Main Result

Proof:(continued)

1. In the latter case when $p \mid M$, then $(M, pp_2p_3\ldots, p_k) \geq p$, contradicts $(M, pp_2, \ldots, p_k) = 1$.
2. In both cases, we get a contradiction.
3. So our assumption in the beginning of the claim is incorrect. Hence $(n^2, p) = 1$ for all $p \leq y_x$ and p is prime.

Proof:(continued)

1. In the latter case when $p \mid M$, then $(M, pp_2 p_3 \ldots, p_k) \geq p$, contradicts $(M, pp_2, \ldots, p_k) = 1$.
2. In both cases, we get a contradiction.
3. So our assumption in the beginning of the claim is incorrect. Hence $(n^2, p) = 1$ for all $p \leq y_x$ and p is prime.
4. Thus we proved the claim.

Proof of Main Result

So now we have: *Proof:*(continued)

1. **Main Claim:**
$$\mathscr{A}_x \subset \mathcal{B}_x$$
2. Thus
$$|\mathscr{A}_x| \leq |\mathcal{B}_x|$$
3. with the notation
$$\mathcal{B}(x) := \lim_{x\to\infty} \frac{|\mathcal{B}_x|}{x}$$
4. and note that
$$\mathscr{A}(x) = \lim_{x\to\infty} \frac{|\mathscr{A}_x|}{x}$$
is the natural density of $\mathscr{A}$.

Proof of Main Result

Proof:(continued)

1. from the fact we just had

$$|\mathscr{A}_x| \leq |\mathcal{B}_x|$$

2. we deduce that

$$\mathscr{A}(x) \leq \mathcal{B}(x)$$

Proof of Main Result

Proof:(continued)

1. Thus we will now estimate $\mathcal{B}(x)$.
2. Claim
$$\mathcal{B}(x) = \mathcal{O}(\frac{1}{\log\log\log x})$$

Proof of Main Result

Proof:(continued)

1. By Erthostenes-Legendre sieve method (Theorem 1.1. in [3]), We have
$$|\mathcal{B}_x| \ll \prod_{p<y_x}(1-\frac{1}{p})$$
2. By Mertens's etimate:
$$\prod_{p<y_x}(1-\frac{1}{p}) \ll \frac{x}{\log y_x} \ll \frac{x}{\log\log\log x}$$
3. Thus
$$|\mathcal{B}_x| \ll \frac{x}{\log\log\log x}$$

Proof:(continued)

1. and so

$$\frac{|\mathcal{B}_x|}{x} \ll \frac{1}{\log\log\log x}$$

2. That is, there exits a constant $c_1 > 0$ such that for large x,

$$\frac{|\mathcal{B}_x|}{x} \leq \frac{c_1}{\log\log\log x}$$

3. taking the limit x going to infinity on both sides:

$$(0 \leq) \lim_{x\to\infty} \frac{|\mathcal{B}_x|}{x} \leq \lim_{x\to\infty} \frac{c_1}{\log\log\log x} = 0$$

4. Thus

$$\mathcal{B}(x) = \lim_{x\to\infty} \frac{|\mathcal{B}_x|}{x} = 0$$

Proof of Main Result

Proof:(continued)

1. By Squeeze Theorem,

$$0 \leq \mathscr{A}(x) \leq \mathcal{B}(x) = 0$$

   gives

2. $$\mathscr{A}(x) = 0$$

3. Thus $\mathscr{A}$ has natural density zero. And we finished the proof.

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References

Paul A. Loomis, **New families of solitary numbers**, Journal of Algebra and Its Applications, Vol. 14, No. 9 (2015).

Florian Luca, **On the densities of some subsets of integers**, Missouri J. Math. Sci. Volume 19, Issue 3 (2007), 167-170.

H. Halberstam and H.E. Rickert, **Sieve Methods**, Academic Press, London, 1974.

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