The Joy of Sets

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See Norm Matloff's web page <http://heather.cs.ucdavis.edu/~matloff/beamer.html> for a quick tutorial.

Disclaimer: Our slides don't show off what Beamer can do. Sorry.

Goals

To define the natural numbers as sets, and to show that these "numbers" behave just like the natural numbers we know and love.

To define the integers from the natural numbers, and the rational numbers from the integers.

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Question: What is wrong with $\mathcal{C}$?
(If you already know the answer, *shhhh*.)

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Definition: A set is a collection of objects.

Question: Is this a good definition?

Consider: $\mathcal{C} = \{A \colon A \text{ is a set and } A \notin A\}$

Question: What is wrong with $\mathcal{C}$?
(If you already know the answer, *shhhh*.)

Think: Either (i) $\mathcal{C} \in \mathcal{C}$ or (ii) $\mathcal{C} \notin \mathcal{C}$.

(i) If $\mathcal{C} \in \mathcal{C}$, then $\mathcal{C} \notin \mathcal{C}$. ⋇

(ii) If $\mathcal{C} \notin \mathcal{C}$, then $\mathcal{C} \in \mathcal{C}$. ⋇

Conclusion: The collection $\mathcal{C}$ leads to contradictions.

A town's barber

The barber in a town, who is a man, cuts the hair of every man in the town who does not cut his own hair.

Who cuts the barber's hair?

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The collection $\mathcal{C} = \{A\colon A \text{ is a set and } A \notin A\}$ and the town's barber are examples of the *Russell paradox*.

The introduction of axioms into set theory was to avoid difficulties such as the Russell paradox.

A very brief history

- Georg Cantor (1895) defined *set* (German: *Menge*) intuitively to be a "collection ... of definite and distinguishable objects. . . ." Also introduced notion of *cardinal numbers* to describe size of infinite sets, but was not very satisfactory.
- Leopold Kronecker (1823–1891) objected strenuously to Cantor's uninhibited use of infinite sets and his analysis.
- Gottlob Frege (1884), logician, suggested definition of cardinal number, but did not become widely known.
- Bertrand Russell, in *Principles of Arithmetic* (1903), gave independently same definition of cardinal number as Frege, which did become widely known.

- Ernst Zermelo (1908) published first successful axiomatic treatment of set theory.
  - two undefined terms (*set* and *membership*)
  - only seven axioms
- Abraham Fraenkel (1920s) and others amended the axioms of Zermelo. Now called *Zermelo-Fraenkel axioms* (ZF) for set theory.
- Zermelo unable to prove internal consistency of his axioms. Kurt Gödel (1930s) published incompleteness theorem that proved ZF (among other logical systems) cannot be proved consistent within itself.
- Many players, e.g., Frege, Peano, Cohen, others.

Axioms of ZF

Objects Sets $x,\ y,\ z, \ldots$

Relation Membership, $x \in y$

So, we have only sets, sets of sets, sets of sets of sets, and so on.

Modeled after intuitive properties of "collections."

"Pack of wolves" and "bunch of grapes" are not actually sets. Sets are abstract notions.

In the axiomatic method: formulate axioms, and deduce (prove) theorems by logical deduction without reference to any interpretation of what appears in the axioms.

E.g., Hilbert (1899) axiomatized Euclidean geometry. In his axiomatization, points/lines/planes may be replaced by tables/chairs/beer mugs and everything carries through.

Axiom systems may have interpretations (models) that are different from the intended ones.

Axiom 1 (Existence) There exists a set with no objects.

Axiom 2 (Extension) Two sets are equal if they contain the same objects.

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Axiom 2 (Extension) Two sets are equal if they contain the same objects.

Proposition There exists only one set that does not contain any objects.

Proof Suppose that two sets A and B do not contain any objects. Then, by extension, $A = B$. $\square$

Definition The set that contains no objects is called the *empty set*, denoted $\varnothing$.

Are there any sets that contain objects?

Axiom 3 (Pairing) If x and y are sets, then $\{x, y\}$ is a set.

Are there any sets that contain objects?

Axiom 3 (Pairing) If x and y are sets, then $\{x,\ y\}$ is a set.

Proposition There is a set that is not empty.

Proof The empty set $\varnothing$ is a set. By pairing, $\{\varnothing,\ \varnothing\}$ is a set that is not empty. $\square$

Remark By extension, $\{\varnothing,\ \varnothing\} = \{\varnothing\}$. Thus, we now have a set that contains one object. Moreover, the same argument shows that if x is a set, then $\{x\}$ is a set.

By pairing, then, $\{\varnothing,\ \{\varnothing\}\}$ is a set that contains two objects.

Pairing and extension allows us to create unordered and ordered pairs.

$\{x, \; y\}$ is an un*ordered* pair. For, by extension, $\{x, \; y\} = \{u, \; v\}$ implies

$$x = u \text{ and } y = v \quad \text{or} \quad x = v \text{ and } y = u.$$

Pairing and extension allows us to create unordered and ordered pairs.

$\{x,\ y\}$ is an <u>un</u>*ordered* pair. For, by extension, $\{x,\ y\} = \{u,\ v\}$ implies

$$x = u \text{ and } y = v \quad \text{or} \quad x = v \text{ and } y = u.$$

$\{\{x\},\ \{x,\ y\}\}$ is an *ordered* pair, i.e.,

$$\{\{x\},\ \{x,\ y\}\} = \{\{u\},\ \{u,\ v\}\}$$

if and only if

$$x = u \quad \text{and} \quad y = v.$$

Think:

$$\langle x, y\rangle = \langle u, v\rangle \quad \text{if and only if} \quad x = u \text{ and } y = v.$$

The "if" direction is straightforward. We prove the "only if" direction.

By extension,

- if $x = y$, then

$$\begin{aligned} & \{\{x\},\ \{x,\ y\}\} = \{\{u\},\ \{u,\ v\}\} \\ \Longrightarrow\quad & \{\{x\}\} = \{\{u\},\ \{u,\ v\}\} \\ \Longrightarrow\quad & \{x\} = \{u\} = \{u,\ v\} \\ \Longrightarrow\quad & x = u = v, \end{aligned}$$

so $x = u$ and $y = v$;

- if $x \neq y$, then

$$\begin{aligned} & \{\{x\},\ \{x,\ y\}\} = \{\{u\},\ \{u,\ v\}\} \\ \Longrightarrow\quad & \{x\} = \{u\} \text{ and } \{x,\ y\} = \{u,\ v\} \\ \Longrightarrow\quad & x = u \\ \Longrightarrow\quad & \{u,\ y\} = \{u,\ v\} \\ \Longrightarrow\quad & y = v, \end{aligned}$$

so $x = u$ and $y = v$.

Definition $\langle x, y\rangle = \{\{x\},\ \{x,\ y\}\}$:

$$\langle x, y\rangle = \langle u, v\rangle \quad \text{if and only if} \quad x = u \text{ and } y = v$$

Is there a set that contains three objects?

Pairing is not enough. For, let x, y, and z be sets. Then,

$$\text{pairing} \implies \{x,\ y\} \text{ is a set, and}$$

$$\text{pairing} \implies \{\{x,\ y\},\ z\} \text{ is a set;}$$

but $\{\{x,\ y\},\ z\}$ has only two objects.

We need another axiom to answer the question, "Is there a set that contains three objects?"

Axiom 4 (Union) If x is a set, then the collection of all the elements of the members of x is a set.

Proposition There is a set that contains three elements.

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Proposition There is a set that contains three elements.

Proof From before, $x = \varnothing$, $y = \{\varnothing\}$, and $z = \{\varnothing, \{\varnothing\}\}$ are three sets. Now,

$$\begin{aligned} \text{pairing} &\implies \{x, y\} \text{ and } \{z\} \text{ are sets;} \\ \text{pairing} &\implies \{\{x, y\}, \{z\}\} \text{ is a set;} \\ \text{union} &\implies \{x, y, z\} \text{ is a set} \end{aligned}$$

that contains three objects. We write $\{x, y\} \cup \{z\} = \{x, y, z\}$.

Remark We may obtain sets with n objects for any natural number n in a similar way.

Now, return to the Russell paradox: $y = \{z \colon z \notin z\}$. The following axiom lets us avoid the Russell paradox.

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Axiom 5 (Separation)   Let x be a set and let $\phi(z)$ be a formula with free variable z (i.e., z is not quantified). The collection of members of x that satisfy $\phi(z)$ is a set:

$$y = \{z \in x \colon \phi(z)\} \quad \text{is a set.}$$

Remark   What separation says is that *if we have a set to begin*, *then separating* apart members of that set according to an unquantified condition gives a set.

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Remark What separation says is that *if we have a set to begin, then separating* apart members of that set according to an unquantified condition gives a set. E.g.,

$$y = \{z \in x\colon z \notin z\}$$

is a set by separation provided x is a set. Separation avoids the Russell paradox.

Since we have defined an ordered pair $\langle x, y \rangle = \{\{x\},\ \{x,\ y\}\}$, we might define a Cartesian product of sets X and Y to be

$$X \times Y = \{\langle x, y \rangle \colon x \in X \text{ and } y \in Y\},$$

Since we have defined an ordered pair $\langle x, y\rangle = \{\{x\},\ \{x,\ y\}\}$, we might define a Cartesian product of sets X and Y to be

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but this does not respect separation. (Russell paradox alert!) To define a Cartesian product properly, we need the following axiom.

Axiom 6 (Power) There is a set of all the subsets of a set.

Note that this set of all the subsets of a set is unique by extension.

Definition The *power set* of a set x, denoted $\mathcal{P}(x)$, contains all of the subsets of x.

Note that $x \in X$ and $y \in Y$ imply

$$\{x\},\ \{x,\ y\} \in \mathcal{P}(X \cup Y),$$

so that

$$\langle x, y \rangle = \{\{x\},\ \{x,\ y\}\} \in \mathcal{P}(\mathcal{P}(X \cup Y)).$$

Note that $x \in X$ and $y \in Y$ imply

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Now we may define the Cartesian product of sets X and Y properly to be

$$X \times Y = \{\langle x, y\rangle \in \mathcal{P}(\mathcal{P}(X \cup Y)) \colon x \in X \text{ and } y \in Y\}.$$

Because $\mathcal{P}(\mathcal{P}(X \cup Y))$ is a set by power, now $X \times Y$ is a set by separation.

With the Cartesian product of sets X and Y, we can now define relation and function between X and Y.

Definition A *relation* R between sets X and Y is a subset of $X \times Y$:

$$R = \{\langle x, y\rangle \in X \times Y \colon x \text{ is related to } y\}.$$

Definition A *function* $f\colon X \to Y$ with domain X and codomain Y is a relation between X and Y such that for every $x \in X$ there is a unique $y \in Y\colon\ f(x) = y$.

Finally, we come to defining the natural numbers.

Recall that existence, extension, pairing, and union allow us to have sets like

$$\varnothing,\ \{\varnothing\},\ \{\varnothing,\ \{\varnothing\}\},\ \{\varnothing,\ \{\varnothing\},\ \{\varnothing,\ \{\varnothing\}\}\},\ldots .$$

Finally, we come to defining the natural numbers.

Recall that existence, extension, pairing, and union allow us to have sets like

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We would like to define

$$0 = \varnothing,$$
$$1 = \{\varnothing\},\quad 2 = \{\varnothing,\ \{\varnothing\}\},\quad 3 = \{\varnothing,\ \{\varnothing\},\ \{\varnothing,\ \{\varnothing\}\}\},\ \ldots,$$

and then show that these "numbers" behave just like the natural numbers we know and love.

What do we need to accomplish this?

We need the notions of “successor” and “inductive set.”

Definition If x is a set, the *successor* of x is defined to be

$$x^+ = x \cup \{x\}.$$

Definition A set x is an *inductive set* if

1. $\varnothing \in x$, and
2. if $y \in x$, then $y^+ \in x$.

The idea is that $\mathbb{N}$ is an inductive set because $0 \in \mathbb{N}$, and $n \in \mathbb{N}$ implies $n^+ = n + 1 \in \mathbb{N}$.

$$
\begin{aligned}
0 &= \varnothing, \\
1 &= 0^+ = \varnothing \cup \{\varnothing\} = \{\varnothing\}, \\
2 &= 1^+ = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing,\ \{\varnothing\}\}, \\
3 &= 2^+ = \{\varnothing,\ \{\varnothing\}\} \cup \{\{\varnothing,\ \{\varnothing\}\}\} \\
&\qquad = \{\varnothing,\ \{\varnothing\},\ \{\varnothing,\ \{\varnothing\}\}\}, \\
4 &= 3^+ = \cdots,
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$$

but there is no guarantee that we can collect them all into one set. For this, we need the following axiom.

Axiom 7 (Infinity) There exists an inductive set.

Before continuing, we define the intersection of sets to be

$$x \cap y = \{z \in x \cup y \colon z \in x \text{ and } z \in y\}.$$

Because $x \cup y$ is a set by union, now $x \cap y$ is a set by separation.

Proposition If x and y are inductive sets, then $x \cap y$ is an inductive set.

Proof

1. Since x and y are inductive sets, $\varnothing \in x$ and $\varnothing \in y$; thus, $\varnothing \in x \cap y$.
2. If $z \in x \cap y$, then $z \in x$ so that $z^+ \in x$, and $z \in y$ so that $z^+ \in y$; thus, $z^+ \in x \cap y$.

Therefore, $x \cap y$ is an inductive set.

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Proof

1. Since x and y are inductive sets, $\varnothing \in x$ and $\varnothing \in y$; thus, $\varnothing \in x \cap y$.
2. If $z \in x \cap y$, then $z \in x$ so that $z^+ \in x$, and $z \in y$ so that $z^+ \in y$; thus, $z^+ \in x \cap y$.

Therefore, $x \cap y$ is an inductive set.

Definition The set of natural numbers $\mathbb{N}$ is defined to be the intersection of all inductive sets.

Thus, $\mathbb{N} = \{0,\ 1,\ 2,\ 3, \ldots\}$ is the smallest inductive set, where

$$0 = \varnothing,\ 1 = 0^+,\ 2 = 1^+,\ 3 = 2^+,\ \ldots .$$

Before we show that $\mathbb{N}$ behaves just like the natural numbers that we know and love, we round out ZF with the following axioms without comment, except regarding "choice."

Axiom 8 (Replacement) The range of a class function restricted to a set is a set.

Axiom 9 (Foundation) Every set contains an ε-minimal element.

Axiom 10 (Choice) If $\mathcal{F}$ is a nonempty family of nonempty sets, then there exists a function $h\colon \mathcal{F} \to \bigcup \mathcal{F}$ such that $h(x) \in x$ for every $x \in \mathcal{F}$.

Remark

- The axiom of choice is controversial because it is not constructive—it simply says that a choice function exists.
- We may consider ZF (without choice) or ZF(C) (with choice).
- Assuming that ZF is consistent,
    - Gödel (1940) proved ZF(C) + CH is consistent;
    - Cohen (1963) proved ZF(C) + $\neg$CH is consistent.

(CH = continuum hypothesis)

The natural numbers, $\mathbb{N}$

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the smallest inductive set, where

$$\begin{aligned} 0 &= \varnothing, \\ 1 &= 0^+ = \varnothing \cup \{\varnothing\} = \{\varnothing\}, \\ 2 &= 1^+ = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}, \\ 3 &= 2^+ = \{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\} \\ &= \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\ 4 &= 3^+ = \cdots . \end{aligned}$$

Inductive set: $0 \in \mathbb{N}$, and $n \in \mathbb{N}$ implies $n^+ = n + 1 \in \mathbb{N}$

Order The elements of $\mathbb{N}$ are ordered, $<$ and $\leq$, using $\in$.

For $m, \ n \in \mathbb{N}$, define

- $m < n$ if $m \in n$;
- $m \leq n$ if $m < n$ or $m = n$.

E.g., $1 = \{\varnothing\}$ and $3 = \{\varnothing, \ \{\varnothing\}, \ \{\varnothing, \ \{\varnothing\}\}\}$ so that

$$1 \in 3 \quad \Longrightarrow \quad 1 < 3.$$

Similarly, $0 < 3$ and $2 < 3$.

Indeed, can show that $(\mathbb{N}, \in)$ is well-ordered, i.e., every subset of $\mathbb{N}$ has a least element.

Arithmetic We define $+$ and $\cdot$ by recursion.

For $m, \ n \in \mathbb{N}$, define

- $m + 0 = m$;
- $m + n^+ = (m + n)^+$.

E.g., $1 + 3 = 1 + 2^+ = \underbrace{\boxed{(1 + 2)^+ = 3^+}}_{\text{by recursion}} = 4$.

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Define

- $m \cdot 0 = 0$;
- $m \cdot n^+ = m \cdot n + m$.

E.g., $1 \cdot 3 = 1 \cdot 2^+ = \underbrace{\boxed{1 \cdot 2 + 1 = 2 + 1}}_{\text{by recursion}} = 3.$

With these definitions of $+$ and $\cdot$, $(\mathbb{N}, +, \cdot)$ is isomorphic to the usual natural numbers. Hence, can show that addition and multiplication are commutative, and so on—all of the properties of addition and multiplication of natural numbers.

E.g.,

$$1 + 3 = 1 + 2^+ = (1 + 2)^+ = 3^+ = 4$$

and

$$3 + 1 = 3 + 0^+ = (3 + 0)^+ = 3^+ = 4,$$

so that

$$1 + 3 = 3 + 1.$$

The integers, $\mathbb{Z}$

Consider the integer -2: it is the result of $3-5$, $7-9$, and so on. This motivates us to define -2 to be the *ordered pair* $\langle m, n\rangle$ if $m - n = -2$. So, e.g.,

$$-2 = \langle 3, 5\rangle \quad \text{and} \quad -2 = \langle 7, 9\rangle,$$

and so on; that is, $\langle 3, 5\rangle$ and $\langle 7, 9\rangle$ are "equivalent integers" because both ordered pairs represent -2.

But this does not make sense yet because we do not have negative numbers yet. Negative numbers (negative integers) are precisely what we are trying to define from the natural numbers.

Begin with $\mathbb{N}$. Define on $\mathbb{N} \times \mathbb{N}$ the equivalence relation $\sim_{\mathbb{Z}}$ by

$$\langle m, n \rangle \sim_{\mathbb{Z}} \langle k, l \rangle \quad \text{if} \quad m + l = n + k.$$

E.g., $\langle 3, 5 \rangle \sim_{\mathbb{Z}} \langle 7, 9 \rangle$ because $3 + 9 = 5 + 7$. Also,

$$\langle 3, 5 \rangle \sim_{\mathbb{Z}} \langle 0, 2 \rangle \quad \text{and} \quad \langle 7, 9 \rangle \sim_{\mathbb{Z}} \langle 0, 2 \rangle.$$

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$$\langle 3, 5 \rangle \sim_{\mathbb{Z}} \langle 0, 2 \rangle \quad \text{and} \quad \langle 7, 9 \rangle \sim_{\mathbb{Z}} \langle 0, 2 \rangle.$$

In fact, every $\langle m, n \rangle$ can be "reduced" to $\langle a, 0 \rangle$ or $\langle 0, b \rangle$ in "lowest terms."

Indeed, think $\langle a, 0 \rangle = a$ and $\langle 0, b \rangle = -b$; that is, think

$$a = \{ \langle m, n \rangle \in \mathbb{N} \times \mathbb{N} \colon \langle m, n \rangle \sim_{\mathbb{Z}} \langle a, 0 \rangle \},$$
$$-b = \{ \langle m, n \rangle \in \mathbb{N} \times \mathbb{N} \colon \langle m, n \rangle \sim_{\mathbb{Z}} \langle 0, b \rangle \}.$$

These are equivalence classes.

Now define

$$\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/ \sim_{\mathbb{Z}}$$

with order, $+$, and $\cdot$ as follows.

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with order, $+$, and $\cdot$ as follows.

Order $\langle m, n \rangle < \langle k, l \rangle$ if $m + l < n + k$

Think $\langle 3, 6 \rangle = -3$ and $\langle 7, 9 \rangle = -2$, and note that

$$3 + 9 < 6 + 7 \quad \text{so that} \quad \langle 3, 6 \rangle < \langle 7, 9 \rangle;$$

that is, $-3 < -2$. Define $\leq$ similarly.

Addition $\langle m, n\rangle + \langle k, l\rangle = \langle m + k, n + l\rangle$

E.g.,

$$\underset{-3}{\langle 3, 6\rangle} + \underset{-2}{\langle 7, 9\rangle} = \underset{-5}{\langle 10, 15\rangle}$$

Addition $\langle m, n\rangle + \langle k, l\rangle = \langle m+k, n+l\rangle$

E.g.,

$$\underset{-3}{\langle 3, 6\rangle} + \underset{-2}{\langle 7, 9\rangle} = \underset{-5}{\langle 10, 15\rangle}$$

Multiplication $\langle m, n\rangle \cdot \langle k, l\rangle = \langle mk+nl, ml+nk\rangle$

E.g.,

$$\underset{-3}{\langle 3, 6\rangle} \cdot \underset{-2}{\langle 7, 9\rangle} = \underset{6}{\langle 75, 69\rangle}$$

The rational numbers, $\mathbb{Q}$

Define on $\mathbb{Z} \times \mathbb{Z}^+$ the equivalence relation $\sim_{\mathbb{Q}}$ by

$$\langle a, b \rangle \sim_{\mathbb{Q}} \langle c, d \rangle \quad \text{if} \quad ad = bc.$$

The rational numbers, $\mathbb{Q}$

Define on $\mathbb{Z} \times \mathbb{Z}^+$ the equivalence relation $\sim_\mathbb{Q}$ by

$$\langle a, b \rangle \sim_\mathbb{Q} \langle c, d \rangle \quad \text{if} \quad ad = bc.$$

- In the ordered pair $\langle a, b \rangle$, both a and b are themselves ordered pairs as we have defined $\mathbb{Z}$; that is,

$$\langle a, b \rangle \ = \langle \langle m, n \rangle, \langle k, l \rangle \rangle,$$

  where $\langle m, n \rangle, \langle k, l \rangle \in \mathbb{Z}$.

- $\mathbb{Z}^+$ means

$$\mathbb{Z} \setminus \{ \langle m, n \rangle \in \mathbb{N} \times \mathbb{N} \colon \langle m, n \rangle \sim_\mathbb{Z} \langle 0, 0 \rangle \};$$

  that is, $\mathbb{Z}^+$ is $\mathbb{Z}$ minus the equivalence class of $\langle 0, 0 \rangle$.

E.g., $\langle -4, 6\rangle \sim_{\mathbb{Q}} \langle -10, 15\rangle$ because $(-4)(15) = (6)(-10)$. Also,

$$\langle -4, 6\rangle \sim_{\mathbb{Q}} \langle -2, 3\rangle \quad \text{and} \quad \langle -10, 15\rangle \sim_{\mathbb{Q}} \langle -2, 3\rangle.$$

In fact, every $\langle a, b\rangle$ can be reduced to $\langle p, q\rangle$ in lowest terms; i.e., p and q have no common factors besides 1 or -1.

Indeed, think $\langle p, q\rangle = \dfrac{p}{q}$.

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Now define

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^{+}) / \sim_{\mathbb{Q}}$$

with order, $+$, and $\cdot$ as follows.

Order $\langle a, b\rangle < \langle c, d\rangle$ if $ad < bc$

Think $\dfrac{a}{b} < \dfrac{c}{d}$ if $ad < bc$. Define $\leq$ similarly.

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Think $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$.

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Addition $\langle a, b\rangle + \langle c, d\rangle = \langle ad + bc, bd\rangle$

Think $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$.

Multiplication $\langle a, b\rangle \cdot \langle c, d\rangle = \langle ac, bd\rangle$

Think $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$.

Real numbers, $\mathbb{R}$, and complex numbers, $\mathbb{C}$

With the rational numbers in hand, we may define the real numbers using *Dedekind cuts* or *Cauchy sequences* of rational numbers with order, $+$, and $\cdot$ defined appropriately.

Then, with the real numbers in hand, we may define the complex numbers as ordered pairs of real numbers with $+$, and $\cdot$ defined appropriately.

Thank you

References

1. Paul R. Halmos, *Naive Set Theory*.
2. Patrick Suppes, *Axiomatic Set Theory*.
3. Walter R. Rudin, *Principles of Mathematical Analysis*.

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