Mathematics and Television

Michael Behrens

California State University, Los Angeles

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Outline

• The CRT Screen
• The YIQ Color Space
• Image Processing
• The NTSC Signal
• The RGB Gamut vs. The YIQ Gamut
The CRT Screen

- A CRT television draws an image one horizontal line at a time. Each line is called a *scanline*.
- An incoming signal controls the intensity of a beam that traces a path along each scanline, lighting up colored phosphors.
- The intensity of the beam affects the resultant brightness of each phosphor.
Black & White vs. Color Television

• For a black and white television, there is only 1 signal involved, which translates to the brightness (in shades of grey) of each part of the image.
• For a color television, there are 3 signals (red, green, and blue). Each signal affects the phosphors of the corresponding color.
• When viewed from a distance, the red, green, and blue light blends together to create the final color of each part of the image.
From Black & White to Color

• One of the challenges faced when developing color television was that it had to be backwards-compatible with the existing black & white television sets.

• How could one design a signal that was comprehensible to both black & white and color TV sets?

• The answer to this problem was found in the YIQ color space.
The YIQ Color Space

- Y is the *brightness* component. By itself, the Y signal gives black-and-white television.
- I and Q are the *color* components. They are orthogonal basis vectors for the 2D color space.
Component Bounds

- The YIQ color space is used with:
  \[
  Y \in [0, 1] \\
  I \in [-0.5957, 0.5957] \\
  Q \in [-0.5226, 0.5226]
  \]

- The RGB color space is used with:
  \[
  R \in [0, 1] \\
  G \in [0, 1] \\
  B \in [0, 1]
  \]
The IQ Plane

$Y = 0.5, I \in [-0.6, 0.6], Q \in [-0.6, 0.6]$
The conversion from RGB to YIQ is a linear transformation, so it can be represented by a matrix multiplication.

\[
\begin{bmatrix}
Y \\
I \\
Q
\end{bmatrix} =
\begin{bmatrix}
0.299 & 0.587 & 0.114 \\
0.596 & -0.274 & -0.322 \\
0.211 & -0.523 & 0.312
\end{bmatrix}
\begin{bmatrix}
R \\
G \\
B
\end{bmatrix}
\]
The I and Q values can be viewed as the components of a vector in the IQ plane. 
The magnitude of the vector is the color’s saturation. 
The angle the vector makes with the horizontal axis is the color’s hue.
Some common image processing adjustments are easier to perform in the YIQ color space than in the RGB color space.
The hue of a color can be changed by rotating the IQ vector by $\theta$ degrees.

\[
\begin{bmatrix}
    Y_2 \\
    I_2 \\
    Q_2
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & \cos \theta & \sin \theta \\
    0 & -\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
    Y_1 \\
    I_1 \\
    Q_1
\end{bmatrix}
\]
Hue

\[ \theta = -\frac{\pi}{9} \text{ (or } -20^\circ) \]
Hue

\[ \theta = \frac{\pi}{9} \text{ (or } 20^\circ) \]
Saturation

The saturation of a color can be changed by scaling the IQ vector by the factor $\alpha \geq 0$.

\[
\begin{bmatrix}
Y_2 \\
I_2 \\
Q_2
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & \alpha
\end{bmatrix}
\begin{bmatrix}
Y_1 \\
I_1 \\
Q_1
\end{bmatrix}
\]
Saturation

\[ \alpha = 0.4 \]
Saturation

\[ \alpha = 1.6 \]
Brightness and Contrast

- The \textit{black level} of an image is the lowest possible $Y$ value.
- The \textit{white level} of an image is the highest possible $Y$ value.
- The \textit{brightness} setting usually adjusts the black level, and the \textit{contrast} setting adjusts the difference between the two levels.
Brightness and Contrast

A given $Y$ value (with $Y \in [0, 1]$) can be remapped to the interval $[B, W]$, where $B$ is the black level and $W$ is the white level, as follows:

\[
\begin{bmatrix}
Y_2 \\
l_2 \\
Q_2
\end{bmatrix} = \begin{bmatrix}
W - B & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}\begin{bmatrix}
Y_1 \\
l_1 \\
Q_1
\end{bmatrix} + \begin{bmatrix}
B \\
0 \\
0
\end{bmatrix}
\]

Note, of course, that $B, W \in [0, 1]$ and $B < W$. 
$B = 0.0, W = 0.95$
$B = 0.1, W = 0.95$
Contrast

$B = 0.05, W = 0.9$
Contrast

\[ B = 0.05, \quad W = 1.0 \]
The NTSC Signal

The image portion of the NTSC signal has the following form:

\[ f(t) = Y(t) + I(t) \sin(\omega_c t) + Q(t) \cos(\omega_c t) \]

- \( Y(t) \), \( I(t) \), \( Q(t) \) are the YIQ components as functions of time.
- \( \omega_c = 2\pi f_c \), with \( f_c = 3579545 \) Hz (this is called the NTSC colorburst frequency).
An Alternative Form

Note that the color information can also be viewed as a single sine wave, instead of as the sum of a sine and cosine wave. This gives us the form:

\[ f(t) = Y(t) + S(t) \sin(\omega_c t + H(t)) \]

where \( S(t) \) is the saturation and \( H(t) \) is the hue.
I, Q, and the Combined Color Signal

I is in Red, Q is in Blue, and the Combined Signal is in Green
Derivation of the Alternative Form

Using the Sine Addition Formula:

\[ \sin(x + y) = \sin x \cos y + \cos x \sin y \]

We can rewrite the alternative form as follows:

\[
f(t) = Y(t) + S(t)\left[\sin(\omega_c t) \cos H(t) + \cos(\omega_c t) \sin H(t)\right]
\]

\[
f(t) = Y(t) + S(t) \cos H(t) \sin(\omega_c t) + S(t) \sin H(t) \cos(\omega_c t)
\]
Derivation of the Alternative Form

Thus, the two forms are equivalent if:

\[ I(t) = S(t) \cos H(t) \quad \text{and} \quad Q(t) = S(t) \sin H(t) \]

Earlier, we noted that \( I \) and \( Q \) were orthogonal basis vectors, and that the \( IQ \) vector determined the color information. The magnitude of the \( IQ \) vector was the saturation, and the angle between the \( IQ \) vector and the horizontal axis was the hue.
Derivation of the Alternative Form

The magnitude of the IQ vector is:

\[
\sqrt{(I(t))^2 + (Q(t))^2} = \sqrt{(S(t) \cos H(t))^2 + (S(t) \sin H(t))^2} \\
= S(t) \sqrt{\cos^2 H(t) + \sin^2 H(t)} \\
= S(t)
\]

Thus, we have that the magnitude is \( S(t) \).
Derivation of the Alternative Form

The angle of the IQ vector is:

\[
\tan \theta = \frac{Q(t)}{I(t)} \rightarrow \tan \theta = \frac{S(t) \sin H(t)}{S(t) \cos H(t)}
\]

\[
\rightarrow \tan \theta = \tan H(t)
\]

Thus, we have that the angle is \( H(t) \).
Decomposing the NTSC Signal

The NTSC signal must be decomposed back into the Y, I, and Q component signals before being converted back to RGB and sent to the CRT screen.
Separating Y & IQ

- The I and Q signals are sine waves with a frequency of 3579545 Hz.
- A *bandpass* filter centered at this frequency, when applied to the YIQ signal, will isolate the IQ (color) signal.
Consider the color components of the signal:

\[ f(t) = I(t) \sin(\omega_c t) + Q(t) \cos(\omega_c t) \]

If this signal is multiplied by \( \sin(\omega_c t) \), we have the following:

\[ f(t) \sin(\omega_c t) = I(t) \sin^2(\omega_c t) + Q(t) \sin(\omega_c t) \cos(\omega_c t) \]
Separating I & Q

Using the Pythagorean Identity \((\sin^2 \theta + \cos^2 \theta = 1)\) and the Sine Double Angle Formula \((\sin(2\theta) = 2\sin \theta \cos \theta)\), this becomes:

\[
f(t) \sin(\omega_c t) = I(t)(1 - \cos^2(\omega_c t)) + Q(t)\left(\frac{1}{2} \sin(2\omega_c t)\right)
\]

\[
f(t) \sin(\omega_c t) = I(t) - I(t) \cos^2(\omega_c t) + \frac{1}{2} Q(t) \sin(2\omega_c t)
\]

If a lowpass filter is applied to this signal, the second two terms will vanish, leaving the isolated \(I(t)\) signal.
Similarly, multiplying the original $f(t)$ by $\cos(\omega_c t)$ gives:

$$f(t) \cos(\omega_c t) = Q(t) - Q(t) \sin^2(\omega_c t) + \frac{1}{2} l(t) \sin(2\omega_c t)$$

Again, a lowpass filter applied to this signal will give the isolated $Q(t)$ signal.
Artifacts & Color Bleed

- The separation methods discussed here are imperfect.
- Part of the IQ signal remains in the Y signal, and part of the Y signal remains in the IQ signal, leading to artifacts.
- Similarly, part of the I signal remains in the Q signal, and vice versa, leading to color bleed.
Bandwidth Limits

- Furthermore, the lowpass filters employed in this discussion impose a limit on the bandwidths of the Y, I, and Q signals.
- This limits the maximum frequency present in the separated signal, which translates to how fast the value of that signal can change when moving along a scanline (i.e., its horizontal resolution).
- A lower bandwidth limit leads to blurring in the horizontal direction.
Component Video

- The Red, Green, and Blue signals all travel along their own cables.
- No separation is necessary, and no artifacts are present.
- The bandwidth is only limited by the cable itself.
Component Video
S-Video

• The Y signal and IQ signal have their own cables.
• There are no artifacts from the mixing of those two signals, but color bleed is still present, as the I and Q signals still need to be separated from each other.
• The bandwidth of the I and Q signals is also limited.
S-Video
Composite Video

- The video signal is a combined YIQ signal on a single cable. This is analogous to the NTSC signal.
- Artifacts and color bleed are present, and the bandwidth of the Y, I and Q signals is limited.
Composite Video
Converting from YIQ to RGB

\[
\begin{bmatrix}
  R \\
  G \\
  B \\
\end{bmatrix} =
\begin{bmatrix}
  1.0 & 0.956 & 0.619 \\
  1.0 & -0.272 & -0.647 \\
  1.0 & -1.106 & 1.703 \\
\end{bmatrix}
\begin{bmatrix}
  Y \\
  I \\
  Q \\
\end{bmatrix}
\]

Note that this matrix is the inverse of the matrix used to convert from RGB to YIQ.
The RGB Gamut vs. The YIQ Gamut

• When converting from YIQ to RGB, it is possible for the resultant RGB value to be outside of $[0, 1] \times [0, 1] \times [0, 1]!$
• This is because YIQ has a larger gamut than RGB.
RGB Clipping

- One solution is to clip the resultant RGB values to the interval $[0, 1]$.
- However, doing so will change the hue of the color, which may be undesirable.
Saturation Clipping

- Another solution is to desaturate the color until it lies within the RGB gamut.
- Recall that desaturating a color involves scaling the IQ vector.
- What does desaturation look like in RGB?
YIQ to RGB Revisited

The matrix multiplication shown previously can also be expressed as:

\[
\begin{bmatrix}
R \\
G \\
B
\end{bmatrix} = Y \begin{bmatrix}
1.0 \\
1.0 \\
1.0
\end{bmatrix} + I \begin{bmatrix}
0.956 \\
-0.272 \\
-1.106
\end{bmatrix} + Q \begin{bmatrix}
0.619 \\
-0.647 \\
1.703
\end{bmatrix}
\]

Desaturating the color involves multiplying both \( I \) and \( Q \) by the same scalar \( \alpha \), with \( 0 \leq \alpha \leq 1 \).
YIQ to RGB Revisited

\[
\begin{bmatrix}
R \\
G \\
B
\end{bmatrix} = \begin{bmatrix}
Y \\
Y \\
Y
\end{bmatrix} + \alpha \begin{bmatrix}
0.956 & -0.272 & 0.619 \\
-0.272 & 0.813 & -0.715 \\
0.619 & -0.715 & 1.703
\end{bmatrix}
\]

- If \( \alpha = 0 \), the color is completely desaturated, and the resultant RGB value is a shade of grey with luminance \( Y \).
- What value of \( \alpha \) should be selected so that the RGB value lies within the gamut?
YIQ to RGB Revisited

We can linearly interpolate between the shade of grey with luminance Y and the original RGB value as follows:

$$\begin{bmatrix} R_2 \\ G_2 \\ B_2 \end{bmatrix} = \alpha \begin{bmatrix} R_1 \\ G_1 \\ B_1 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} Y \\ Y \\ Y \end{bmatrix}$$

This can be rewritten as:

$$\begin{bmatrix} R_2 \\ G_2 \\ B_2 \end{bmatrix} = \alpha \begin{bmatrix} R_1 \\ G_1 \\ B_1 \end{bmatrix} + \begin{bmatrix} Y \\ Y \\ Y \end{bmatrix} - \alpha \begin{bmatrix} Y \\ Y \\ Y \end{bmatrix}$$

Source: "Efficient gamut clipping for color image processing using LHS and YIQ" by Christopher C. Yang and S. H. Kwok, Optical Engineering Vol. 42 No. 3, March 2003
YIQ to RGB Revisited

\[
\begin{bmatrix}
R_2 \\
G_2 \\
B_2
\end{bmatrix}
- \begin{bmatrix}
Y \\
Y \\
Y
\end{bmatrix}
= \alpha \begin{bmatrix}
R_1 \\
G_1 \\
B_1
\end{bmatrix}
- \alpha \begin{bmatrix}
Y \\
Y \\
Y
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_2 - Y \\
G_2 - Y \\
B_2 - Y
\end{bmatrix}
= \alpha \begin{bmatrix}
R_1 - Y \\
G_1 - Y \\
B_1 - Y
\end{bmatrix}
\]

Considering this as three separate equations, we have that:

\[
\alpha = \frac{R_2 - Y}{R_1 - Y}, \quad \alpha = \frac{G_2 - Y}{G_1 - Y}, \quad \alpha = \frac{B_2 - Y}{B_1 - Y}
\]

Source: "Efficient gamut clipping for color image processing using LHS and YIQ” by Christopher C. Yang and S. H. Kwok, Optical Engineering Vol. 42 No. 3, March 2003
Desaturating a Color

Let’s assume that the red component \( R > 1 \). In this case, we want to desaturate the color so that the red component becomes 1. Plugging in \( R_1 = R \) and \( R_2 = 1 \), we have that:

\[
\alpha = \frac{1 - Y}{R - Y}
\]

Source: "Efficient gamut clipping for color image processing using LHS and YIQ" by Christopher C. Yang and S. H. Kwok, Optical Engineering Vol. 42 No. 3, March 2003
Desaturating a Color

Similarly, if $G > 1$ or $B > 1$, we have that:

$$\alpha = \frac{1 - Y}{G - Y} \text{ or } \alpha = \frac{1 - Y}{B - Y}$$

Source: "Efficient gamut clipping for color image processing using LHS and YIQ" by Christopher C. Yang and S. H. Kwok, Optical Engineering Vol. 42 No. 3, March 2003
Desaturating a Color

If more than one of the components is greater than 1, we want to take the \textit{smallest} of the possible values for \( \alpha \), giving us:

\[
\alpha = \min\left(\frac{1 - Y}{R - Y}, \frac{1 - Y}{G - Y}, \frac{1 - Y}{B - Y}\right)
\]

Source: ”Efficient gamut clipping for color image processing using LHS and YIQ” by Christopher C. Yang and S. H. Kwok, Optical Engineering Vol. 42 No. 3, March 2003
Desaturating a Color

Note that only the components that are greater than 1 should be included in the expression. For example, if $R > 1$ and $G > 1$, but $0 \leq B \leq 1$, then we want to find:

$$\alpha = \min\left(\frac{1 - Y}{R - Y}, \frac{1 - Y}{G - Y}\right)$$

Source: "Efficient gamut clipping for color image processing using LHS and YIQ" by Christopher C. Yang and S. H. Kwok, Optical Engineering Vol. 42 No. 3, March 2003
Questions?